

Optimal Smooth Portfolio Selection for An Insider

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Abstract

We study the optimal portfolio problem for an insider, in the case that the performance is measured in terms of the logarithm of the terminal wealth minus a term measuring the roughness and the growth of the portfolio. We give explicit solutions in some cases. Our method uses stochastic calculus of forward integrals.

1 Introduction

There has been an increasing interest in the insider trading in recent years (see for example [1]–[6] and [8]–[10] and the references therein). By an *insider* in a financial market we mean a certain investor who possesses more information than the information generated by the financial market itself. An insider may be for example an executive or simply an employee of a company. In probabilistic terminology information is generally represented by a filtration. Usually an investor can only use the filtration generated by the market to make a decision. We call such investors *honest*. An insider has a larger filtration (more information) available to him and can use this larger filtration to make his decision (for example to maximize his portfolio).

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To simplify our presentation we assume that the market consists of the following two assets over the time period $[0, T]$. The first one is a bond whose price is determined by a stochastic process

$$dS_0(t) = r(t)S_0(t)dt, \quad 0 \leq t \leq T.$$

Another asset is the stock whose price follows the following geometric Brownian motion

$$dS_1(t) = S_1(t) [\mu(t)dt + \sigma(t)dB(t)], \quad 0 \leq t \leq T,$$

where $r(t)$, $\mu(t)$, and $\sigma(t)$ are deterministic functions and $B(t) = B_t(\omega)$, $0 \leq t \leq T$, is a Brownian motion and $dB(t)$ denotes the Itô type stochastic differential. Denote $\mathcal{F}_t = \sigma(B_s, 0 \leq s \leq t)$, the information generated by the market. Assume for example that at the beginning ($t = 0$) the insider knows in addition the future value of the underlying Brownian motion at time T_0 , where $T_0 > T$. Then his information filtration is given by $\mathcal{G}_t = \sigma(B_s, 0 \leq s \leq t) \vee \sigma(B_{T_0})$, the filtration generated by the Brownian motion up to time t and B_{T_0} . The insider may use this filtration (rather than as usual use only the filtration \mathcal{F}_t) to optimize his portfolio.

More explicitly, let us express the portfolio in terms of the fraction $\pi(t)$ of the total wealth invested in the stocks at time t . Let $X^{(\pi)}(t)$ denote the corresponding wealth at time t . In [9] the problem of maximizing the expectation of the logarithmic utility of terminal wealth,

$$(1.1) \quad \Phi_{\mathcal{G}} := \sup_{\pi} \left\{ \mathbb{E} \left[\log(X^{(\pi)}(T)) \right] \right\}$$

is considered, where the supremum is taken over all \mathcal{G}_t -adapted portfolios $\pi(\cdot)$. They prove that in this case the optimal insider portfolio is

$$(1.2) \quad \pi^*(t) = \frac{\mu(t) - r(t)}{\sigma^2(t)} + \frac{B(T_0) - B(t)}{\sigma(t)(T_0 - t)}.$$

Moreover, the corresponding maximal expected utility $\Phi_{\mathcal{G}}$ is given by

$$\Phi_{\mathcal{G}} = \mathbb{E} \left[\int_0^T \left\{ r(s) + \frac{1}{2} \frac{(\mu(s) - r(s))^2}{\sigma^2(s)} + \frac{1}{2(T_0 - s)} \right\} ds \right]; \quad T_0 \geq T.$$

In particular, if $T_0 = T$ we get

$$\Phi_{\mathcal{G}} = \infty.$$

This is clearly an unrealistic result. If $T_0 = T$ we see by (1.2) that the optimal portfolio π^* needed to achieve $\Phi_{\mathcal{G}} = \infty$ will converge towards the derivative of $B(t)$ at $t = T_0^-$. Thus $\pi^*(t)$ will consist of more and more wild fluctuations as $t \rightarrow T_0^-$. This is both practically impossible and also undesirable from the point of view of the insider: He does not want to expose a too conspicuous portfolio, compared to that of the honest trader, which in the optimal case is just

$$\pi_{\text{honest}}^*(t) = \frac{\mu(t) - r(t)}{\sigma^2(t)}.$$

To model this constraint we propose to modify the problem (1.1) to the following:

PROBLEM 1.1 Find $\pi^* \in \mathcal{A}_{\mathcal{G}}$ and Φ such that

$$\begin{aligned}\Phi &= \sup_{\pi \in \mathcal{A}_{\mathcal{G}}} \mathbb{E} \left[\log \left(X^{(\pi)}(T) \right) - \int_0^T |\mathbb{Q}\pi(s)|^2 ds \right] \\ &= \mathbb{E} \left[\log \left(X^{(\pi^*)}(T) \right) - \int_0^T |\mathbb{Q}\pi^*(s)|^2 ds \right],\end{aligned}$$

where $\mathcal{A}_{\mathcal{G}}$ is a suitable family of admissible \mathcal{G}_t -adapted portfolios π . Here $\mathbb{Q} : \mathcal{A}_{\mathcal{G}} \rightarrow \mathcal{A}_{\mathcal{G}}$ is some linear operator measuring the size and/or the fluctuations of the portfolio. For example we could have

$$(1.3) \quad \mathbb{Q}\pi(s) = \lambda_1(s)\pi(s),$$

where $\lambda_1(s) \geq 0$ is some given weight function. This models the situation where the insider is penalized for large volumes of trade.

An alternative choice of \mathbb{Q} would be

$$(1.4) \quad \mathbb{Q}\pi(s) = \lambda_2(s)\pi'(s),$$

for some weight function $\lambda_2(s) \geq 0$. ($\pi'(s) = \frac{d}{ds}\pi(s)$.) In this case the insider is penalized for large trade fluctuations. Other choices of \mathbb{Q} are also possible, including combinations of (1.2) and (1.3).

We will return to Problem 1.1 in Section 3, after giving a brief introduction to the forward integral.

2 The Forward Integral

In general $B(t)$ need not be a semimartingale with respect to a bigger filtration $\mathcal{G}_t \supset \mathcal{F}_t$. A simple example is

$$\mathcal{G}_t = \mathcal{F}_{t+\delta}; \quad t \geq 0$$

where $\delta > 0$ is a constant.

Therefore, to be able to deal with corresponding (anticipating) \mathcal{G}_t -adapted integrands $\phi(t, \omega)$, we must go beyond the semimartingale integral context. Following [3] we propose to use *the forward* integral to model such situations. This integral extends the semimartingale integral in the sense that the two integrals coincide if $B(t)$ is a semimartingale with respect to \mathcal{G}_t .

In this section we briefly review some basic concepts and results on forward integrals. We refer to [3] for motivations for using forward integrals in insider trading, and to [12], [13] for more information about forward integrals.

Definition 2.1 ([12]) *Let $\phi(t, \omega)$ be a measurable process (not necessarily adapted). Then the forward stochastic integral of ϕ is defined as*

$$\int_0^\infty \phi(t, \omega) d^- B(t) = \lim_{\varepsilon \rightarrow 0} \int_0^\infty \phi(t, \omega) \frac{B(t + \varepsilon) - B(t)}{\varepsilon} dt$$

if the convergence is in probability.

Let $\pi : 0 = t_0 < t_1 < \dots < t_n = t$ be a partition of $[0, T]$ and denote $|\pi| = \max_{0 \leq j \leq n-1} (t_{j+1} - t_j)$. It is easy to see that if ϕ is càdlàg, then

$$(2.1) \quad \int_0^T \phi(t, \omega) d^- B(t) = \lim_{|\pi| \rightarrow 0} \sum_{j=0}^{n-1} \phi(t_j) (B(t_{j+1}) - B(t_j)).$$

(See [3] for details.)

Definition 2.2 *By a (1-dimensional) forward process we mean a process $X(t) = X(t, \omega)$ of the form*

$$(2.2) \quad X(t) = x + \int_0^t u(s, \omega) ds + \int_0^t v(s, \omega) d^- B(s); \quad t > 0,$$

where $u(s, \omega)$ and $v(s, \omega)$ are measurable processes (not necessarily \mathcal{F}_t -adapted) such that

$$\int_0^t |u(s, \omega)| ds < \infty, \quad \text{a.s. for all } t > 0$$

and the Itô forward integral

$$\int_0^t v(s, \omega) d^- B(s)$$

exists for all $t > 0$.

In accordance with the classical Itô process notation, we use the short hand notation

$$d^- X(t) = u(t)dt + v(t)d^- B(t)$$

for the integral equation (2.2).

Theorem 2.3 ([13]) *(An Itô formula for forward processes) Let*

$$d^- X(t) = u(t)dt + v(t)d^- B(t)$$

be a forward process. Let $f \in C^2(\mathbb{R})$ and define

$$Y(t) = f(X(t)).$$

Then $Y(t)$ is also a forward process and

$$d^- Y(t) = f'(X(t))d^- X(t) + \frac{1}{2}f''(X(t))v^2(t)dt$$

As an application of the Itô formula for forward integrals we get

Corollary 2.4 ([3]) *Let $u(t)$, $v(t)$ be measurable processes such that the integrals*

$$\int_0^t (|u(s)|^2 + |v(s)|^2) ds \quad \text{and} \quad \int_0^t v(s) d^- B(s) \quad \text{exist for all } t > 0.$$

Then the forward stochastic differential equation

$$dX(t) = X(t) \left[u(t) dt + v(t) d^- B(t) \right] ; \quad X(0) = x > 0$$

has the unique solution

$$X(t) = x \exp \left(\int_0^t \left(u(s) - \frac{1}{2} v^2(s) \right) ds + \int_0^t v(s) d^- B(s) \right).$$

We also need the following, which follows easily from the definition:

Lemma 2.5 *Suppose $\phi(t)$ is forward integrable and G is an \mathcal{F}_T -measurable random variable. Then*

$$\int_0^T G \phi(t) d^- B(t) = G \int_0^T \phi(t) d^- B(t).$$

3 Optimal Smooth Portfolio for An Insider

We now return to Problem 1.1 in the introduction. So we assume the market consists of the two investment possibilities:

(i) A bond, with price given by

$$dS_0(t) = r(t) S_0(t) dt ; \quad S_0(0) = 1 ; \quad 0 \leq t \leq T.$$

(ii) A stock, with price given by

$$dS_1(t) = S_1(t) [\mu(t) dt + \sigma(t) dB(t)] ; \quad 0 \leq t \leq T,$$

where $T > 0$ is constant and $r(t)$, $\mu(t)$ and $\sigma(t)$ are given \mathcal{F}_t -adapted processes. We assume that

$$\mathbb{E} \left[\int_0^T \left\{ |\mu(t)| + |r(t)| + \sigma^2(t) \right\} dt \right] < \infty$$

$$\sigma(t) \neq 0 \quad \text{for a.a. } (t, \omega) \in [0, T] \times \Omega.$$

Let $\mathcal{G}_t \supset \mathcal{F}_t$ be the information filtration available to the insider and let $\pi(t)$ be the portfolio chosen by the insider, measured in terms of the fraction of the total wealth $X(t) = X^{(\pi)}(t)$ invested in the stock at time $t \in [0, T]$. Then the corresponding wealth $X(t) = X^{(\pi)}(t)$ at time t is modeled by the forward differential equation

$$\begin{aligned} dX(t) &= (1 - \pi(t)) X(t) r(t) dt + \pi(t) X(t) [\mu(t) dt + \sigma(t) d^- B(t)] \\ (3.1) \quad &= X(t) \left[[r(t) + (\mu(t) - r(t)) \pi(t)] dt + \sigma(t) \pi(t) d^- B(t) \right]. \end{aligned}$$

For simplicity we assume $X(0) = 1$. The motivation for using this forward integral model for the anticipating stochastic differential equation (3.1) is the formula (2.1), which expresses the forward integral as a limit of Riemann sums of the Itô type, i.e. where the i -th term has the form $\phi(t_i)(B(t_{i+1}) - B(t_i))$ with ϕ evaluated at the *left* end point t_i of the interval $[t_i, t_{i+1}]$. Moreover, if $B(t)$ happens to be a semimartingale with respect to \mathcal{G}_t , then indeed the forward integral coincides with the semimartingale integral. See [3] and [12], [13] for more details on this.

We now specify the set $\mathcal{A} = \mathcal{A}_{\mathcal{G}}$ of the admissible portfolios π as follows:

Definition 3.1 *In the following we let $\mathcal{A} = \mathcal{A}_{\mathcal{G}}$ denote a linear space of stochastic processes $\pi(t)$ such that (3.2)–(3.5) hold, where*

$$(3.2) \quad \begin{aligned} \pi(t) & \text{ is } \mathcal{G}_t - \text{ adapted and the } \sigma - \text{ algebra generated by } \{\pi(t); \pi \in \mathcal{A}\} \\ & \text{ is equal to } \mathcal{G}_t, \text{ for all } t \in [0, T], \end{aligned}$$

$$(3.3) \quad \pi \text{ belongs to the domain of } \mathbb{Q},$$

$$(3.4) \quad \sigma(t)\pi(t) \text{ is forward integrable,}$$

$$(3.5) \quad \mathbb{E} \left[\int_0^T |\mathbb{Q}\pi(t)|^2 dt \right] < \infty.$$

With these definitions we can now specify Problem 1.1 as follows:

Problem 3.2 *Find Φ and $\pi^* \in \mathcal{A}$ such that*

$$\Phi = \sup_{\pi \in \mathcal{A}} J(\pi) = J(\pi^*),$$

where

$$J(\pi) = \mathbb{E} \left[\log(X^{(\pi)}(T)) - \frac{1}{2} \int_0^T |\mathbb{Q}\pi(s)|^2 ds \right],$$

$\mathbb{Q} : \mathcal{A} \rightarrow \mathcal{A}$ being a given linear operator (\mathbb{E} denotes the expectation with respect to P). We call Φ the value of the insider and $\pi^* \in \mathcal{A}$ an optimal portfolio (if it exists).

We now proceed to solve Problem 3.2: Using Corollary 2.4 we get that the solution of (3.1) is

$$\begin{aligned} X(t) = & \exp \left(\int_0^t \left\{ r(s) + (\mu(s) - r(s))\pi(s) - \frac{1}{2}\sigma^2(s)\pi^2(s) \right\} ds \right. \\ & \left. + \int_0^t \sigma(s)\pi(s)d^-B(s) \right). \end{aligned}$$

Therefore we get

$$(3.6) \quad \begin{aligned} J(\pi) = & \mathbb{E} \left[\int_0^T \left\{ r(t) + (\mu(t) - r(t))\pi(t) - \frac{1}{2}\sigma^2(t)\pi^2(t) \right\} dt \right. \\ & \left. + \int_0^T \sigma(t)\pi(t)d^-B(t) - \frac{1}{2} \int_0^T |\mathbb{Q}\pi(t)|^2 dt \right]. \end{aligned}$$

To maximize $J(\pi)$ we use a calculus of variation technique, as follows: Suppose an optimal insider portfolio $\pi = \pi^*$ exists (in the following we omit the $*$). Let $\theta \in \mathcal{A}$ be another portfolio. Then the function

$$f(y) := J(\pi + y\theta); \quad y \in \mathbb{R}$$

is maximal for $y = 0$ and hence

$$(3.7) \quad \begin{aligned} 0 &= f'(0) = \frac{d}{dy} [J(\pi + y\theta)]_{y=0} \\ &= \mathbb{E} \left[\int_0^T \left\{ (\mu(t) - r(t))\theta(t) - \sigma^2(t)\pi(t)\theta(t) \right\} dt \right. \\ &\quad \left. + \int_0^T \sigma(t)\theta(t)d^-B(t) - \int_0^T \mathbb{Q}\pi(t)\mathbb{Q}\theta(t)dt \right]. \end{aligned}$$

Let \mathbb{Q}^* denote the adjoint of \mathbb{Q} in the Hilbert space $L^2([0, T] \times \Omega)$, namely,

$$\mathbb{E} \left[\int_0^T \alpha(t)(\mathbb{Q}\beta)(t)dt \right] = \mathbb{E} \left[\int_0^T (\mathbb{Q}^*\alpha)(t)\beta(t)dt \right]$$

for all α and β in \mathcal{A} . Then we can rewrite (3.7) as

$$(3.8) \quad \mathbb{E} \left[\int_0^T \left\{ \mu(t) - r(t) - \sigma^2(t)\pi(t) - \mathbb{Q}^*\mathbb{Q}\pi(t) \right\} \theta(t)dt + \int_0^T \sigma(t)\theta(t)d^-B(t) \right] = 0.$$

Now we apply this to a special choice of θ : Fix $t \in [0, T]$ and $h > 0$ such that $t + h < T$ and choose

$$\theta(s) = \theta_0(t)\chi_{[t, t+h]}(s); \quad s \in [0, T],$$

where $\theta_0(t)$ is \mathcal{G}_t -measurable. Then by Lemma 2.5 we have

$$\begin{aligned} \mathbb{E} \left[\int_0^T \sigma(s)\theta(s)d^-B(s) \right] &= \mathbb{E} \left[\int_t^{t+h} \sigma(s)\theta_0(t)d^-B(s) \right] \\ &= \mathbb{E} \left[\theta_0(t) \int_t^{t+h} \sigma(s)dB(s) \right]. \end{aligned}$$

Combining this with (3.8) we get

$$\mathbb{E} \left[\left(\int_t^{t+h} \left\{ \mu(s) - r(s) - \sigma^2(s)\pi(s) - \mathbb{Q}^*\mathbb{Q}\pi(s) \right\} ds + \int_t^{t+h} \sigma(s)dB(s) \right) \theta(t) \right] = 0.$$

Since this holds for all such $\theta(t)$ we conclude that

$$\mathbb{E} [M(t+h) - M(t)|\mathcal{G}_t] = 0,$$

where

$$(3.9) \quad M(t) := \int_0^t \left\{ \mu(s) - r(s) - \sigma^2(s)\pi(s) - \mathbb{E} [\mathbb{Q}^*\mathbb{Q}\pi(s)|\mathcal{G}_s] \right\} ds + \int_0^t \sigma(s)dB(s).$$

Since $\sigma \neq 0$ this proves

Theorem 3.3 *Suppose an optimal insider portfolio $\pi \in \mathcal{A}$ for Problem 3.2 exists. Then*

$$dB(t) = d\hat{B}(t) - \frac{1}{\sigma(t)} \left\{ \mu(t) - \rho(t) - \sigma^2(t)\pi(t) - \mathbb{E} [\mathbb{Q}^*\mathbb{Q}\pi(t)|\mathcal{G}_t] \right\} dt$$

where $\hat{B}(t) := \int_0^t \sigma^{-1}(s)dM(s)$ is a \mathcal{G}_t -Brownian motion. In particular,

$$B(t) \text{ is a semimartingale with respect to } \mathcal{G}_t.$$

We now use this to find an equation for an optimal portfolio π :

Theorem 3.4 *Assume that there exists a process $\gamma_t(s, \omega)$ such that $\gamma_t(s)$ is \mathcal{G}_t -measurable for all $s \leq t$ and*

$$t \rightarrow \int_0^t \gamma_t(s)ds \text{ is of finite variation a.s.}$$

and

$$(3.10) \quad N(t) := B(t) - \int_0^t \gamma_t(s)ds \text{ is a martingale with respect to } \mathcal{G}_t.$$

Assume that $\pi \in \mathcal{A}$ is optimal. Then

$$(3.11) \quad \sigma^2(t)\pi(t) + \mathbb{E} [\mathbb{Q}^*\mathbb{Q}\pi(t)|\mathcal{G}_t] = \mu(t) - r(t) + \sigma(t) \frac{d}{dt} \left(\int_0^t \gamma_t(s)ds \right).$$

Proof By comparing (3.9) and (3.10) we get that

$$\sigma(t)dN(t) = dM(t),$$

i.e.

$$-\sigma(t) \frac{d}{dt} \left(\int_0^t \gamma_t(s)ds \right) = \mu(t) - r(t) - \sigma^2(t)\pi(t) - \mathbb{E} [\mathbb{Q}^*\mathbb{Q}\pi(t)|\mathcal{G}_t].$$

□

Next we turn to a partial converse of Theorem 3.4:

Theorem 3.5 *Suppose (3.10) holds. Let $\pi(t)$ be a process solving the equation (3.11). Suppose $\pi \in \mathcal{A}$. Then π is optimal for Problem 3.2.*

Proof Substituting

$$dB(t) = dN(t) + \frac{d}{dt} \left(\int_0^t \gamma_t(s) ds \right) dt$$

and

$$\sigma(t)\pi(t)d^-B(t) = \sigma(t)\pi(t)dN(t) + \sigma(t)\pi(t)\frac{d}{dt} \left(\int_0^t \gamma_t(s) ds \right) dt$$

into (3.6) we get

$$(3.12) \quad J(\pi) = \mathbb{E} \left[\int_0^T \left\{ r(t) + (\mu(t) - r(t))\pi(t) - \frac{1}{2}\sigma^2(t)\pi^2(t) + \sigma(t)\pi(t)\frac{d}{dt} \left(\int_0^t \gamma_t(s) ds \right) - \frac{1}{2}|\mathbb{Q}\pi(t)|^2 \right\} dt \right].$$

This is a concave functional of π , so if we can find $\pi = \pi^* \in \mathcal{A}$ such that

$$\frac{d}{dy} [J(\pi^* + y\theta)]_{y=0} = 0 \quad \text{for all } \theta \in \mathcal{A},$$

then π^* is optimal. By a computation similar to the one leading to (3.8) we get

$$\begin{aligned} \frac{d}{dy} [J(\pi^* + y\theta)]_{y=0} &= \mathbb{E} \left[\int_0^T \left\{ \mu(t) - r(t) - \sigma^2(t)\pi^*(t) + \sigma(t)\frac{d}{dt} \int_0^t \gamma_t(s) ds - \mathbb{Q}^*\mathbb{Q}\pi(t) \right\} \theta(t) dt \right]. \end{aligned}$$

This is 0 if $\pi = \pi^*$ solves equation (3.11). □

We now apply this to some examples:

Example 3.6 Choose

$$(3.13) \quad \mathbb{Q}\pi(t) = \lambda_1(t)\sigma(t)\pi(t)$$

where $\lambda_1(t) \geq 0$ is deterministic.

Then (3.11) takes the form

$$\sigma^2(t)\pi(t) + \lambda_1^2(t)\sigma^2(t)\pi(t) = \mu(t) - r(t) + \sigma(t)\frac{d}{dt} \int_0^t \gamma_t(s) ds$$

or

$$(3.14) \quad \pi(t) = \pi^*(t) = \frac{\mu(t) - r(t) + \sigma(t)\frac{d}{dt} \int_0^t \gamma_t(s) ds}{\sigma^2(t)[1 + \lambda_1^2(t)]}.$$

Substituting this into the formula (3.12) for $J(\pi)$ we obtain

Theorem 3.7 Suppose (3.10) and (3.13) hold. Let $\pi^*(t)$ be given by (3.14). If $\pi \in \mathcal{A}$ then π^* is optimal for Problem 3.2. Moreover, the insider value is

$$(3.15) \quad \begin{aligned} \Phi &= J(\pi^*) \\ &= \mathbb{E} \left[\int_0^T \left\{ r(t) + \frac{1}{2}(1 + \lambda_1^2(t))^{-1} \left(\frac{\mu(t) - r(t)}{\sigma(t)} + \frac{d}{dt} \int_0^t \gamma_t(s) ds \right)^2 \right\} dt \right]. \end{aligned}$$

In particular, if we consider the case mentioned in the introduction, where

$$\mathcal{G}_t = \mathcal{F}_t \vee \sigma(B(T_0)) \quad \text{for some } T_0 > T,$$

then, by a result of Itô [7],

$$\gamma_t(s) = \gamma(s) = \frac{B(T_0) - B(s)}{T_0 - s}$$

and (3.14) becomes

$$\pi^*(t) = \sigma^{-2}(t) [1 + \lambda_1^2(t)]^{-1} \left[\mu(t) - r(t) + \frac{\sigma(t)}{T_0 - t} (B(T_0) - B(t)) \right].$$

The corresponding value is, by (3.15),

$$J(\pi^*) = \mathbb{E} \left[\int_0^T \left\{ r(t) + \frac{1}{2}(1 + \lambda_1^2(t))^{-1} \left(\frac{\mu(t) - r(t)}{\sigma(t)} + \frac{B(T_0) - B(t)}{T_0 - t} \right)^2 \right\} dt \right].$$

In particular, we see that if $\sigma(t) \geq \sigma_0 > 0$ and

$$(3.16) \quad \lambda_1(t) = (T_0 - t)^{-\beta} \quad \text{for some constant } \beta > 0,$$

then

$$J(\pi^*) \leq C_1 + C_2 \int_0^T (T_0 - t)^{-1+2\beta} dt < \infty,$$

even if $T_0 = T$.

Thus if we penalize large investments near $t = T_0$ according to (3.16) the insider gets a finite value even if $T_0 = T$.

Example 3.8 Next we put

$$(3.17) \quad \mathbb{Q}\pi(t) = \pi'(t) \quad (= \frac{d}{dt}\pi(t)).$$

This means that the insider is being penalized for large portfolio fluctuations. Choose \mathcal{A} to be the set of all continuously differentiable processes $\pi(t)$ satisfying (3.2)–(3.5) and in addition

$$(3.18) \quad \pi(0) = \pi(T) = 0 \quad a.s.$$

For simplicity assume that

$$\sigma(t) \equiv 1.$$

Then (3.11) gets the form

$$\pi(t) - \pi''(t) = a(t),$$

where

$$a(t) = \mu(t) - r(t) + \frac{d}{dt} \left(\int_0^t \gamma_t(s) ds \right).$$

Using the *variation of parameter* method we obtain the solution

$$(3.19) \quad \pi(t) = \int_0^t \sinh(t-s) a(s) ds + K \sinh(t),$$

where, as usual, $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$, $x \in \mathbb{R}$, is the hyperbolic sinus function and the constant K is chosen such that $\pi(T) = 0$. In particular, if we again consider the case

$$\mathcal{G}_t = \mathcal{F}_t \vee \sigma(B(T_0)), \quad T_0 > T,$$

so that

$$\gamma_t(s) = \gamma(s) = \frac{B(T_0) - B(s)}{T_0 - s}, \quad 0 \leq s \leq T.$$

we obtain, by (3.19),

$$(3.20) \quad \pi(t) = \int_0^t \sinh(t-s) \left[\mu(s) - r(s) + \frac{B(T_0) - B(s)}{T_0 - s} \right] ds + K \sinh(t).$$

The corresponding value is by (3.12),

$$\begin{aligned} J(\pi) = \mathbb{E} & \left[\int_0^T \left\{ r(t) + (\mu(t) - r(t)) \pi(t) - \frac{1}{2} \pi^2(t) \right. \right. \\ & \left. \left. + \pi(t) \frac{B(T_0) - B(t)}{T_0 - t} - \frac{1}{2} (\pi'(t))^2 \right\} dt \right]. \end{aligned}$$

Note that if $0 \leq t \leq T < T_0$ then

$$\begin{aligned} \mathbb{E} \left[\pi(t) \frac{B(T_0) - B(t)}{T_0 - t} \right] & \leq \mathbb{E} \left[\int_0^t \sinh(t-s) \frac{(B(T_0) - B(s))(B(T_0) - B(t))}{(T_0 - s)(T_0 - t)} ds \right] \\ & = \int_0^t \sinh(t-s) \frac{ds}{T_0 - s}. \end{aligned}$$

Therefore

$$J(\pi) \leq \int_0^T \left(\int_0^t \sinh(t-s) \frac{ds}{T_0 - s} \right) dt \leq \int_0^T \frac{\cosh(T-s) - 1}{T-s} ds \quad \text{for all } T_0 > T.$$

We have proved:

Theorem 3.9 *Suppose $\mathbb{Q}\pi(t) = \pi'(t)$ and \mathcal{A} is chosen as in (3.17), (3.18) and assume that $\sigma(t) = 1$. Then the optimal insider portfolio is given by (3.19). In particular, if we choose*

$$\mathcal{G}_t = \mathcal{F}_t \vee \sigma(B(T_0)) \quad \text{with } T_0 > T,$$

then the optimal portfolio π is given by (3.20) and the corresponding insider value $J(\pi)$ is uniformly bounded for $T_0 > T$.

Remark 3.10 *Both of Examples 3.6 and 3.8 yield ways to penalize the insider investors so that he would not obtain infinite utility. In Example 3.6, $\lambda_1(t) = (T_0 - t)^{-\beta}$ for some $\beta > 0$. To use this penalization, one needs to know T_0 . In Example 3.8, T_0 is not required to be known.*

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